BRANCHING PROCESSES AND KOENIGS FUNCTION

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An explicit solution of time-homogeneous pure birth branching processes is described. It gives alternative extensions for the negative binomial distribution (branching processes with immigration) and for the Furry-Yule distribution (branching processes without immigration).

1 Introduction

The Furry-Yule (FY) and negative binomial (NB) distributions are widely used in phenomenological studies of multiplicity distributions at high energies. Recently, several generalizations of the negative binomial distribution based on the perturbative quantum chromodynamics have appeared in the literature (see for example [1,2]). The aim of this note is to present another generalization [3,4] of the NB and FY distributions based on simple time-homogeneous branching processes,[5,6,7,8] more simple than the branching processes used in QCD. The NB and FY distributions occur in time-homogeneous branching processes with allowed transition $1 \to 2$ and one can extend them allowing higher order transitions $1 \to n$ with $n > 2$. It may happen that these processes can be useful in the description of the hadronization stages still not fully understood in the framework of the QCD.

In section 2, we describe the distribution [3,4] for the pure birth branching process with allowed higher-order transitions that can be used as an extension of the FY distribution. In section 3, the distribution for the branching process with immigration [4] is presented. A discussion is given in the last section.

2 Distributions for general pure birth branching processes

A branching process with continuous evolution parameter t is determined by the rates α_n for the transition ("splitting") of one particle into n particles with all particles subsequently evolving independently. For a pure birth branching process $\alpha_0 = 0$. The FY distribution occurs when only the α_2 is not equal to zero. The probability generating function $m(x,t) = \sum_{n=0}^{\infty} p_n(t) x^n$ satisfies the forward Kolmogorov equation[6,7] $\frac{\partial m}{\partial t} = f(x) \frac{\partial m}{\partial x}$ with $f(x) = \sum_{n=2}^{\infty} \alpha_n x^n - \alpha x$ and $\alpha = \sum \alpha_n$. The Taylor expansion of this equation leads to the following

system of equations for the probabilities p_n

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = -\alpha p_1 \qquad , \tag{1}$$

and for $n > 1$

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = \sum_{j=1}^{n-1} j\alpha_{n-j+1}p_j - n\alpha p_n \qquad . \tag{2}$$

For the case of N initial particles with $p_n^{(N)}(0) = \delta_{Nn}$, the solution of this system has the following form:[4]

$$p_n^{(N)} = \sum_{j=N}^{n} \pi_{jn}^{(N)} p_1^j \ , \tag{3}$$

with the coefficients $\pi_{jn}^{(N)}$ obeying the recursion:

$$(n-j)\pi_{jn}^{(N)} = \sum_{l=1}^{n-j} (n-l)b_l \ \pi_{j(n-l)}^{(N)} \qquad . \tag{4}$$

Here, $p_1 = \exp(-\alpha t)$ and $b_l = \alpha_{l+1}/\alpha$. This recursion starts from $\pi_{NN}^{(N)} = 1$ and the coefficient $\pi_{nn}^{(N)}$ can be found from the relation $\pi_{nn}^{(N)} = -\sum_{j=N}^{n-1} \pi_{jn}^{(N)}$.

One can calculate the coefficients $\pi_{jn}^{(N)}$ using the concept of the Koenigs function (see[9,10] and the references therein). For the branching process starting from N particles, the Koenigs function is defined as the limit:

$$K^{(N)}(x) = K^N(x) = \lim_{n\to\infty} \frac{m^N(x, nt)}{(p_1^N)^n} = \sum_{j=N}^{\infty} \kappa_j^{(N)} x^j = \sum_{j=N}^{\infty} \pi_{Nj}^{(N)} x^j \qquad . \tag{5}$$

The recursion (4) leads to the following recurrence for the coefficients $\kappa_j^{(N)}$, $N = 1, 2, ...;\ j = N+1, N+2, ...$:

$$(j-N)\kappa_j^{(N)} = \sum_{l=1}^{j-N} (j-l)b_l \ \kappa_{(j-l)}^{(N)} \qquad . \tag{6}$$

It is convenient to denote $\kappa_{x+n}^{(x)} = t_n(x)$, then $t_0(x) = 1$ and the polynomials $t_n(x)$ are defined by the recursion:[4]

$$nt_n(x) = \sum_{l=1}^{n} (x+n-l)b_l \ t_{n-l}(x) \qquad . \tag{7}$$

The $\kappa_j^{(N)}$ in terms of $t_n(x)$ is equal to $t_{j-N}(N)$.

The remarkable property of the Koenigs function is that it satisfies the functional Schröder equation $K(m) = p_1 K(x)$. This can be easily obtained from the forward Kolmogorov equation $\mathrm{d}m/\mathrm{d}t = f(m)$. Its integration gives also the expression

$$K(x) = \exp\left(-\alpha \int^x \frac{\mathrm{d}u}{f(u)}\right) \qquad . \tag{8}$$

It is convenient to introduce the function $Q(x)$, the inverse of the Koenigs function. Then, the inversion of the Schröder equation gives the functional relations $m(x,t) = Q(p_1 K(x))$ and $m^N(x,t) = Q^N(p_1 K(x))$. The coefficients of the Taylor expansion for $Q^N(x) = \sum Q_j^{(N)} x^j$ can be found using the following relation: $Q_j^{(N)} = \frac{N}{j}\kappa_{-N}^{(-j)}$. One can show that this relation is equivalent to the Bürmann-Lagrange series for the inverse function (see Appendix B in [11] and the references therein). In terms of $t_n(x)$ it gives: $Q_j^{(N)} = \frac{N}{j} t_{j-N}(-j)$. Finally, comparison of equation (3) with the Taylor expansion in x of the $Q^N(p_1(K(x))$ leads to the following expression[4] for the coefficients $\pi_{jn}^{(N)}$ in terms of $t_n(x)$:

$$\pi_{jn}^{(N)} = Q_j^{(N)} \kappa_n^{(j)} = \frac{N}{j} t_{j-N}(-j) t_{n-j}(j) \qquad . \tag{9}$$

In some cases, the functions $K(x)$ and $Q(x)$ can be found explicitly, for example:
for $b_1 = 1$

$$K(x) = \frac{x}{1-x} \qquad , \qquad Q(x) = \frac{x}{1+x} \qquad ; \tag{10}$$

for $b_N = 1$

$$K(x) = \frac{x}{(1-x^N)^{1/N}} \qquad , \qquad Q(x) = \frac{x}{(1+x^N)^{1/N}} \qquad ; \tag{11}$$

for $b_1 + b_2 = 1$

$$K(x) = \frac{x}{(1-x)^{\frac{1}{1+b_2}} (1+b_2 x)^{\frac{b_2}{1+b_2}}} \qquad , \qquad Q(x) = ? \qquad . \tag{12}$$

In the general case, the expressions for $Q(x)$ and $K(x)$ are quite complicated.

The solution for non-critical branching processes with non-zero α_0 is an infinite series in $\exp(-\alpha' t)$, see [4].

3 Solution for branching processes with immigration

For the branching processes with immigration, there is an additional external source of particles appearing generally in clusters of j particles with the differential rates β_j ($\sum \beta_j = b$). The probability generating function for the process starting with zero particles at $t = 0$ can be written [8,12] as

$$M(x,t) = \exp\left(\int_0^t g(m(x,\tau))\mathrm{d}\tau \right) \tag{13}$$

with

$$g(x) = \sum_{i=1}^{\infty} \beta_i x^i - b \qquad , \tag{14}$$

where $m(x,\tau)$ is the solution for the underlying branching process without immigration. For the underlying pure birth branching process. Equ. (13) leads to the following expression:[4]

$$M(x,t) = \exp\left(-bt\right) \exp\left(\sum_{n=1}^{\infty} C_n(t) x^n \right) \tag{15}$$

with

$$C_n(t) = \sum_{i=1}^{n} \beta_i \sum_{j=i}^{n} \pi_{jn}^{(i)} \frac{1 - p_1^j(t)}{j\alpha} \qquad . \tag{16}$$

Let us denote

$$\exp\left(\sum_{n=1}^{\infty} C_n x^n\right) = 1 + \sum_{n=1}^{\infty} V_n x^n \qquad . \tag{17}$$

Then, the final probability $P_n(t) = \exp\left(-bt\right) V_n(t)$. The coefficients $V_n(t)$ can be calculated using the recursive relation:

$$nV_n = \sum_{j=1}^{n} jC_j V_{n-j} \qquad . \tag{18}$$

This relation is known in combinatorics and is used, for example, in the study of combinants.[13,14,15,16] The coefficients $C_n(t)$ are known as the combinants of the multiplicity distribution $P_n(t)$. It is of interest to note the formal analogy between Equ. (17) and the exponential representation for the $K(x)$ (8).

The branching process with immigration leads to the NB distribution when only β_1, α_2 and α_0 are not equal to zero. The resulting distribution is the Poissonian when only β_1 is non-zero.

4 Discussion

In this note we have described explicit expressions for the probability distributions in the branching processes with the parameters equal to the differential rates α_{i+1} and β_i. These distributions can serve as an extension for the NB and/or FY distributions when at least one of the coefficients with $i > 1$ is non-zero. It seems natural to assume that these parameters have no energy dependence and therefore the energy dependence is concentrated in one parameter $p_1(t) = \exp(-\alpha t)$. It is of interest to note that the exponential dependence on t corresponds to the power-law dependence on the mean multiplicity $\langle n \rangle$, since $\langle n \rangle$ has linear dependence on $\exp(f'(1)t)$.

We have applied the distribution with non-zero α_2 and α_3 to the e^+e^- multiplicity data in the paper.[3] It has been assumed that a fixed number N of sources of particle production is produced at some initial stage of the interaction. These sources develop independently of each other according to the time-homogeneous branching process, producing intermediate neutral clusters. Finally, these clusters decay into a pair of charged or neutral hadrons with probabilities ε or $(1-\varepsilon)$, respectively, as in the Goulianos model.[17] The data at c.m. energies $\sqrt{s}$ above 20 GeV are well described with $N = 7$, with energy dependent parameter p_1 and with the parameter ε fluctuating near 0.677. It has been shown that the parameter p_1 has power-law dependence on $\sqrt{s}$, as noted earlier.[18] It has been also shown that α_3 is consistent with zero, but values of the ratio α_3/α of the order ~ 0.05 are not excluded at the present level of statistics. It is of interest to note that the values of ε near $2/3$ are consistent with zero isospin of the intermediate neutral clusters and are different from the value $\varepsilon = 0.5$ used by K. Goulianos.[17]

It is a challenging task to derive explicit expressions for the polynomials $t_n(x)$ from the recurrence (7). This can be done easily when only one of the b_i is not equal to zero, for example, for $b_1 = 1$ the $t_n(x) = \frac{x(x+1)...(x+n-1)}{n!}$.

It is of interest to look for the singularities in the complex plane of the probability generating functions for the time-homogeneous branching processes (as pursued by I. Dremin [19] and E. De Wolf [20]). These singularities are connected with the singularities of the $K(x)$ and $Q(x)$ and, therefore, with the roots of the equation $f(x) = 0$. For example, for the process with $b_N = 1$, the poles are:

$$x_n = \frac{\exp(\frac{2\pi i n}{N})}{(1-p_1^N)^{1/N}} \tag{19}$$

$(n = 1, 2, ..., N)$. They form a regular polygon and move with t to the circumference with unit radius.

Acknowledgments

I am grateful to S. Hegyi for the opportunity to give this talk and I would like to thank the organizers of the Symposium for providing a beautiful surrounding.

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